QUANTUM MECHANICS
Hwk Solution Set 9
Spin States + bosons and fermions

Solution to Problem 4.55

(**part a**)

$$l(l+1)\hbar^2 = 2\hbar^2, P = 1. \tag{1}$$

(**part b**)

$$m\hbar = 0, P = 1/3 \text{ or } m\hbar = \hbar, P = 2/3. \tag{2}$$

(**part c**)

$$3\hbar^2/4, P = 1. \tag{3}$$

(**part d**)

$$+\hbar/2, P = 1/3 \text{ or } -\hbar/2, P = 2/3. \tag{4}$$

(**part e**) Write the angular part of the wavevector as

$$\frac{1}{\sqrt{3}}|1/2, 1/2\rangle|1, 0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1/2, -1/2\rangle|1, 1\rangle. \tag{5}$$

Then use the $1 \times 1/2$ Clebsch-Gordon table in the book to decompose this state.

[See the examples in the text below the table - first find the m's, that are horizontal - in the table you see horizontally: +1 -1/2 then move horizontally to see the coefficient (i.e. 1/3, take the square root) then move up to get the $|3/2, +1/2\rangle$. This gives the first piece of equation (6) below and the second piece is obtained similarly...continue to get equation(7) by locating: 0 1/2].

$$|1/2, -1/2\rangle|1, 1\rangle = \frac{1}{\sqrt{3}}|3/2, 1/2\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1/2, 1/2\rangle \tag{6}$$

$$|1/2, 1/2\rangle|1, 0\rangle = \frac{\sqrt{2}}{\sqrt{3}}|3/2, 1/2\rangle - \frac{1}{\sqrt{3}}|1/2, 1/2\rangle. \tag{7}$$

Substituting these decompositions gives

$$\frac{1}{\sqrt{3}}|1/2,1/2\rangle|1,0\rangle + \frac{\sqrt{2}}{\sqrt{3}}|1/2,-1/2\rangle|1,1\rangle \tag{8}$$

$$= \frac{2\sqrt{2}}{\sqrt{3}}|3/2,1/2\rangle + \frac{1}{3}|1/2,1/2\rangle. \tag{9}$$

Then we get measurements $(3/2)(3/2+1)\hbar^2 = (15/4)\hbar^2$ or $(1/2)(1/2+1)\hbar^2 = (3/4)\hbar^2$, with probabilities $(\frac{2\sqrt{2}}{\sqrt{3}})^2 = 8/9$ and $1/9$ respectively.
(**part f**) $\hbar/2$ with probability of unity.
(**part g**) Square the wavefunction as given, and note that the cross terms are zero due to the orthogonality of the spin 1/2 unit vectors. We have from the tables in the book (or see hwk 6 and hwk 7) that $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}(r/a)exp(-r/(2a))$ and $Y_1^0 = (3/(4\pi))^{1/2}\cos\theta$ and $Y_1^1 = (3/(8\pi))^{1/2}\sin\theta e^{i\phi}$, Thus

$$|R_{21}|^2((1/3)|Y_1^0|^2 + (2/3)|Y_1^1|^2) = \frac{1}{96\pi a^5}r^2 exp(-r/a). \tag{10}$$

(**part h**) Integrate out the angular variables to obtain

$$|R_{21}|^2(1/3)\int |Y_1^0|^2 \sin^2\theta d\theta d\phi = \frac{1}{3}|R_{21}|^2 = \frac{1}{72a^5}r^2 exp(-r/a). \tag{11}$$

Solution to Problem 5.6

(**part a**) See Eq. 5.19 in the text and solutions from previous hwk for the square well:

$$\langle x\rangle_n = a/2 \tag{12}$$

$$\langle x^2\rangle_n = a^2\left[\frac{1}{3} - \frac{1}{2\pi^2}\left(\frac{1}{n^2}\right)\right] \tag{13}$$

$$\langle(\Delta x)^2\rangle_{n,l} = \langle x^2\rangle_n + \langle x^2\rangle_l - 2\langle x\rangle_n\langle x\rangle_l \tag{14}$$

$$\langle(\Delta x)^2\rangle_{n,l} = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right] \tag{15}$$

(**parts b,c**) Calculate the the overlap function (Eq. 5.20 in the text)

$$\langle x\rangle_{n,l} = \frac{2}{a}\int_0^a dx\; x\sin(n\pi x/a)\sin(l\pi x/a) \tag{16}$$

to find

$$\langle x\rangle_{n,l} = (4a)(nl)\frac{(-1+(-1)^{l+n})}{(l^2-n^2)^2\pi^2}. \tag{17}$$

So using Eq. 5.21 we have

$$\langle(\Delta x)^2\rangle_{n,l\ \pm} = \langle(\Delta x)^2\rangle_{n,l} \mp 2|\langle x\rangle_{n,l}|^2 \tag{18}$$

$$\langle(\Delta x)^2\rangle_{n,l\ \pm} = a^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{l^2}\right)\right] \mp 2\left|(4a)(nl)\frac{(-1+(-1)^{l+n})}{(l^2-n^2)^2\pi^2}\right|^2. \tag{19}$$

Solution to Problem 5.7

(**part a**)

$$\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) \tag{20}$$

(**part b**)

$$\frac{1}{\sqrt{6}}[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) + \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) + \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) \tag{21}$$

$$+\psi_b(x_1)\psi_c(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_b(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_a(x_2)\psi_b(x_3)]. \tag{22}$$

(**part c**)

$$\frac{1}{\sqrt{6}}[\psi_a(x_1)\psi_b(x_2)\psi_c(x_3) - \psi_a(x_1)\psi_c(x_2)\psi_b(x_3) - \psi_b(x_1)\psi_a(x_2)\psi_c(x_3) \tag{23}$$

$$+\psi_b(x_1)\psi_c(x_2)\psi_a(x_3) - \psi_c(x_1)\psi_b(x_2)\psi_a(x_3) + \psi_c(x_1)\psi_a(x_2)\psi_b(x_3)]. \tag{24}$$

The above (part c) can be written as $\frac{1}{\sqrt{3!}}\begin{vmatrix} \psi_a(x_1) & \psi_b(x_1) & \psi_c(x_1) \\ \psi_a(x_2) & \psi_b(x_2) & \psi_c(x_2) \\ \psi_a(x_3) & \psi_b(x_3) & \psi_c(x_3) \end{vmatrix}$ and (part b) can be gotten by making all the negative signs positive.