Model Solution: Homework 8

Physics U603 Summer 1 2006

Hecht 10.2 Those of you who gave a correct algebraic derivation received full credit, the idea was to use an alternate geometric approach based on the phasor diagram in Fig. P10.2.

The radius R of the circle defined by the tips of the phasors can be found by considering the right triangle with hypotenuse R and a side of length $E_0/2$ opposite the angle $\delta/2$, which requires $\sin(\delta/2) = (E_0/2)/R$. The total electric field E is a chord of the circle and can be found by dropping a line from the center of the circle that bisects the angle $N\delta$ and forms the third side of a right triangle with a sides of lengths $E/2$ opposite the bisected angle $N\delta/2$ and hypotenuse R. This construction defines the bisected angle according to $\sin(N\delta/2) = (E/2)/R$. Squaring the ratio

$$\frac{\sin(N\delta/2)}{\sin(\delta/2)} = \frac{E}{E_0}$$

of the expressions for the two triangles defines the intensity ratio

$$\frac{I}{I_0} = \frac{\sin^2(N\delta/2)}{\sin^2(\delta/2)}.$$

Hecht 10.8 Minima in the Fraunhofer pattern occur at angles that satisfy $b\sin\theta = m\lambda$, so the width of the slit is $b = m\lambda_0/\sin\theta = 10\,(1.1522\ \mu\text{m})\,/\sin 6.2° = 106.7\ \mu\text{m}$. If the experiment is conducted under water, where $\lambda = \lambda_0/n$, the tenth minimum appears at an angle defined by $\sin\theta = m\lambda_0/nb$, viz.,

$$\theta = \sin^{-1}\left[\frac{10\,(1.1522\ \mu\text{m})}{1.33\,(106.7\ \mu\text{m})}\right] = 4.66°.$$

Hecht 10.18 There are two parallel rectangular apertures. The cross section of the intensity distribution shown in Fig. P.10.18 matches the functional form

$$I = I_0\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\alpha$$

calculated for two slits of non-negligible width b separated by a distance a, with $\alpha = (ka/2)\sin\theta$ and $\beta = (kb/2)\sin\theta$. The first minimum of $(\sin\beta/\beta)^2$ approximately coincides with the third maximum of $\cos^2\alpha$, so $\sin\theta = \lambda/b = 3\lambda/a$, and the slit separation $a = 3b$ is three times the width of either aperture. The intensity distribution is illustrated in Fig. 10.13b. Notice also that minima in the intensity distribution along the orthogonal direction occur at angular positions roughly a factor of two smaller, indicating that the length of the aperture is approximately twice the width.

Hecht 10.22 In order to find the intensity of the first ring, you must find the value of the argument of the function $2J_1(u)/u$ at its first maximum other than $u = 0$ and then evaluate the function of this argument. One way is to evaluate the Bessel function, and find its maxima, using a mathematical package such as MATLAB. A straightforward approach uses the property given by Hecht on pg. 470, that the extrema of the function $J_1(u)/u$ coincide with zeroes of the second order Bessel function $J_2(u)$, the lowest of which occurs at $u = 5.14$. Fig. 10.23 also indicates this value. The function can be evaluated by interpolation between the values given in Table 10.1 to find $J_1(5.14) = -0.33954$ or by evaluating enough terms in the series expansion

$$\begin{aligned}\frac{2J_1(5.14)}{5.14} &= 1 - \frac{(2.57)^2}{1!2!} + \frac{(2.57)^4}{2!3!} - \frac{(2.57)^6}{3!4!} + \frac{(2.57)^8}{4!5!} - \frac{(2.57)^{10}}{5!6!} + \frac{(2.57)^{12}}{6!7!} - \frac{(2.57)^2}{7!8!} + \cdots \\ &= -0.134\ldots\end{aligned}$$

leading in either case to

$$\frac{I(\theta_{\max})}{I(0)} = \left[\frac{2J_1(5.14)}{5.14}\right]^2 = 0.0175.$$

The peak intensity of the first ring is less than 2% of the intensity at the center of the Airy disk!

Hecht 10.26 The minimum resolvable angular separation

$$\Delta\theta = 1.22\frac{\lambda}{D} = 1.22\frac{550 \times 10^{-9}\ \text{m}}{0.75 \times 10^{-3}\ \text{m}} = 8.95 \times 10^{-4}\ \text{rad},$$

5.3 times worse than the nominal 1.7×10^{-4} rad resolution of the eye, as expected for an aperture diameter 4 mm/0.75 mm= 5.3 times smaller than the diameter of the pupil.

Hecht 10.28 The minimum resolvable angular separation

$$\begin{aligned}\Delta\theta &= 1.22\frac{\lambda}{D} = 1.22\frac{550 \times 10^{-9}\ \text{m}}{5.08\ \text{m}} \\ &= 1.32 \times 10^{-7}\ \text{rad} = 7.57 \times 10^{-6}\ \text{deg} = 0.0272\ \text{arcsec}\end{aligned}$$

is the angular width of the Airy disk. At the distance d of our moon, the telesecope can barely resolve two objects separated by a distance

$$\Delta x = d\Delta\theta = \left(3.844 \times 10^8\ \text{m}\right)\left(1.32 \times 10^{-7}\ \text{rad}\right) = 50.8\ \text{m}.$$

The pupil of the human eye has a much smaller diameter, and can barely resolve objects separated by a distance of

$$\begin{aligned}\Delta x &= d\Delta\theta = 1.22\frac{\lambda d}{D} \\ &= 1.22\frac{\left(550\times 10^{-9}\ \mathrm{m}\right)\left(3.844\times 10^{8}\ \mathrm{m}\right)}{\left(4.00\times 10^{-3}\ \mathrm{m}\right)} = 64.5\ \mathrm{km}\end{aligned}$$

on the moon.

In addition to this enormous improvement in image resolution, the large diameter of the telescope objective favors the detection of faint objects by collecting a larger amount of the incident light.

Hecht 10.33 The condition $d\sin\theta = m\lambda$, with $m = 1$, determines the angular position of the first order line. f the grating has 10,000 lines per cm, the distance between grating lines is $d = 10^{-4}$ cm. It is important to realize that the *small angle approximation does not apply* here: the lines corresponding to the wavelengths $\lambda_1 = 588.9953$ nm and $\lambda_2 = 589.5923$ nm appear at angles $\theta_1 = \sin^{-1}(\lambda_1/d) = \sin^{-1}\left(588.9953\times 10^{-9}\ \mathrm{m}/10^{-6}\ \mathrm{m}\right) = 36.0857^\circ$ and $\theta_2 = \sin^{-1} 0.5895923 = 36.1281^\circ$, respectively. On a screen at distance D, the lines are separated by a distance $\Delta y = D(\tan\theta_2 - \tan\theta_1) = 1.11$ mm.

The small angle approximation should not become an automatic reflex. Here, $\sin\theta = y/D$ is a very poor approximation and seriously underestimates the line separation as $\Delta y = D\Delta\lambda/d = 0.597$ mm!