Model Solution: Homework 1B

Physics U603 Summer 1 2006

1. Substitution of the expressions given for ω_1, ω_2, k_1, and k_2 allows the wave function to be factored as

$$\psi = Ae^{i(kx-\omega t)}\left[e^{-i(\Delta kx-\Delta\omega t)} + e^{i(\Delta kx-\Delta\omega t)}\right].$$

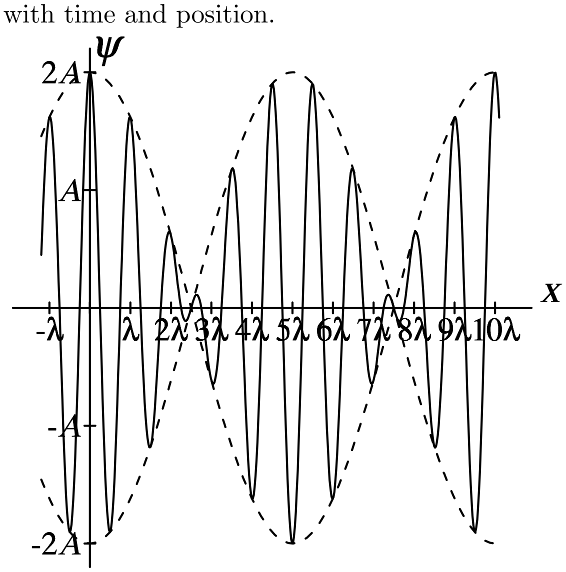
It is simple to show that $e^{i\theta} + e^{-i\theta} = 2\cos\theta$ using Euler's formula for the complex exponential, since the imaginary parts cancel, so the wave function is

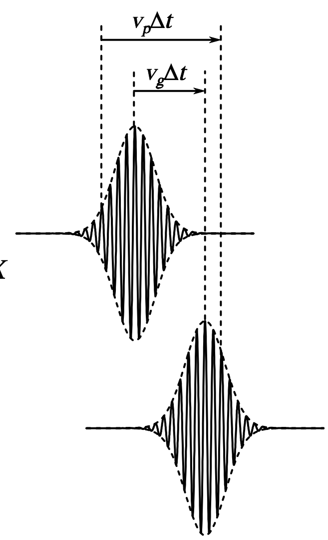
$$\psi = \left[2A\cos\left(\Delta kx - \Delta\omega t\right)\right]e^{i(kx-\omega t)}$$

and we will take the real part

$$\psi = \left[2A\cos\left(\Delta kx - \Delta\omega t\right)\right]\cos\left(kx-\omega t\right)$$

as the physical solution. For $\Delta\omega < \omega$ this can be visualized as a wave of frequency ω with a varying amplitude $2A\cos\left(\Delta kx - \Delta\omega t\right)$ that varies with time and position.





Although the high frequency carrier wave travels at the phase speed $v_p =$

ω/k, as we have seen, the envelope function travels at a rate $v_g = \Delta\omega/\Delta k$ known as the group speed. These two speeds are identical if the phase speed is a constant. In the general case where the phase speed varies with frequency, a wave packet formed by superposing waves with a range of frequencies travels with a group speed $d\omega/dk$, that may differ from the phase speed. For electromagnetic waves, the group speed is the rate at which a packet of energy travels and never exceeds the speed c of light in a vacuum.

2. (a) The product $\theta w_0 = \lambda/\pi$ of the angular divergence θ and minimum beam width λ is limited by the wavelength λ, so $\theta = \lambda/\pi w_0 = \left(633 \times 10^{-9}\text{ m}\right)/\pi\left(0.24 \times 10^{-3}\text{ m}\right) = 840\ \mu\text{rad} = 0.048°$. The realtion $\theta = w/z$ of this value to the beam width w and the distance z from the beam waist determines of the Rayleigh range $2z_0 = 2w_0/\theta = 2\left(0.24 \times 10^{-3}\text{ m}\right)/\left(840 \times 10^{-6}\right) = 0.572$ m and the beam width $w = \theta z = \left(840 \times 10^{-6}\right)\left(3.84 \times 10^{8}\text{ m}\right) = 322$ km at the Moon (about one tenth the diameter of the Moon).

   (b) In spite of the low beam power, the highly directed energy leads to a peak intensity $I = P/\left(\pi w_0^2/2\right) = 2\left(1.3 \times 10^{-3}\text{ W}\right)/\pi\left(0.24 \times 10^{-3}\text{ m}\right)^2 = 14.4\text{ kW/m}^2$ at the beam waist that is much larger than the intensity $I = P/4\pi r^2 = 100\text{ W}/4\pi\left(0.14\text{ m}\right)^2 = 0.406\text{ kW/m}^2$ 14 cm from a much more powerful 100 W isotropic source. Staring directly into a 100 W light bulb 14 cm away for a few seconds should convince you of the importance of using appropriate eye protection even with "weak" laser sources.

3. Bessel beams

   (a) Since the beam profile is a function of s, only derivatives with respect to x and y contribute to the Laplacian

$$\nabla^2 f(s) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

   The chain rule allows the first derivative

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} = \left(\frac{x}{s}\right)\frac{\partial f}{\partial s}$$

   and the second derivative

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{x}{s}\frac{\partial f}{\partial s}\right)\\ &= \frac{x}{s}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial s}\right) + \frac{\partial f}{\partial s}\frac{\partial}{\partial x}\left(\frac{x}{s}\right)\\ &= \left(\frac{x}{s}\right)\frac{\partial^2 f}{\partial s^2}\frac{\partial s}{\partial x} + \left(\frac{s - x\left(\partial s/\partial x\right)}{s^2}\right)\frac{\partial f}{\partial s}\\ &= \left(\frac{x}{s}\right)^2\frac{\partial^2 f}{\partial s^2} + \left(\frac{s^2 - x^2}{s^3}\right)\frac{\partial f}{\partial s}\end{aligned}$$

to be expressed in terms of derivatives with respect to s. Calculating the analogous expression for the other derivative and using $s^2 = x^2 + y^2$ leads to

$$\nabla^2 f(s) = \frac{\partial^2 f}{\partial s^2} + \frac{1}{s}\frac{\partial f}{\partial s}.$$

(b) Substitution of $\psi = f(s)\, e^{i(k_z z - \omega t)}$ into the wave equation yields

$$e^{i(k_z z-\omega t)}\nabla^2 f(s) + f(s)\frac{\partial^2}{\partial z^2}e^{i(k_z z-\omega t)} = \frac{1}{c^2}f(s)\frac{\partial^2}{\partial t^2}e^{i(k_z z-\omega t)}.$$

The derivatives with respect to z and t give factors of ik_z and $-i\omega$, as usual. Substitution of the expression for $\nabla^2 f(s)$ from part (a) leads to

$$\left(\frac{\partial^2 f}{\partial s^2} + \frac{1}{s}\frac{\partial f}{\partial s}\right)e^{i(k_z z-\omega t)} - k_z^2 f e^{i(k_z z-\omega t)} = -\frac{\omega^2}{c^2} f e^{i(k_z z-\omega t)}.$$

After factoring out $e^{i(k_z z-\omega t)}$, rearrangement gives

$$\frac{\partial^2 f}{\partial s^2} + \frac{1}{s}\frac{\partial f}{\partial s} + \left(k_0^2 - k_z^2\right) f = 0,$$

with $k_0 = \omega/c$.

(c) Writing $f(s) = Ay(k_s s)$ in the form of the dimensionless variable $x = k_s s$, the chain rule gives $\partial f/\partial s = (\partial f/\partial x)(\partial x/\partial s) = k_s A y'(x)$ and $\partial^2 f/\partial s^2 = k_s^2 A y''(x)$. Substitution yields

$$k_s^2 A y'' + \frac{k_s}{s} A y' + \left(k_0^2 - k_z^2\right) A y = 0,$$

and, if $k_0^2 = k_s^2 + k_z^2$, Bessel's equation

$$y'' + \frac{1}{x}y' + y = 0,$$

results after factoring out the common factor $k_s^2 A$.

(d) The resulting solution $\psi = AJ_0(k_s s)\, e^{i(k_z z-\omega t)}$ of the wave equation takes the form of a wave travelling in the z direction with an amplitude, and intensity, that varies with transverse distance. The intensity vanishes when the value of the Bessel function drops to zero. According to the table, the smallest displacement where this happens is when $k_s s = 2.4048$. We describe the diameter of the beam as $D = 2s = 4.8096/k_s$.

(e) Unlike a Gaussian beam, a beam with this Bessel profile can travel without diverging. However, because $k_0^2 = k_s^2 + k_z^2$, travelling wave solutions are restricted to values $k_s^2 \le k_0^2 = (2\pi/\lambda)^2$, and the resulting minimum beam width $D \ge 4.8096\lambda/2\pi = 0.485\ \mu\text{m}$ is still limited by the wavelength.