

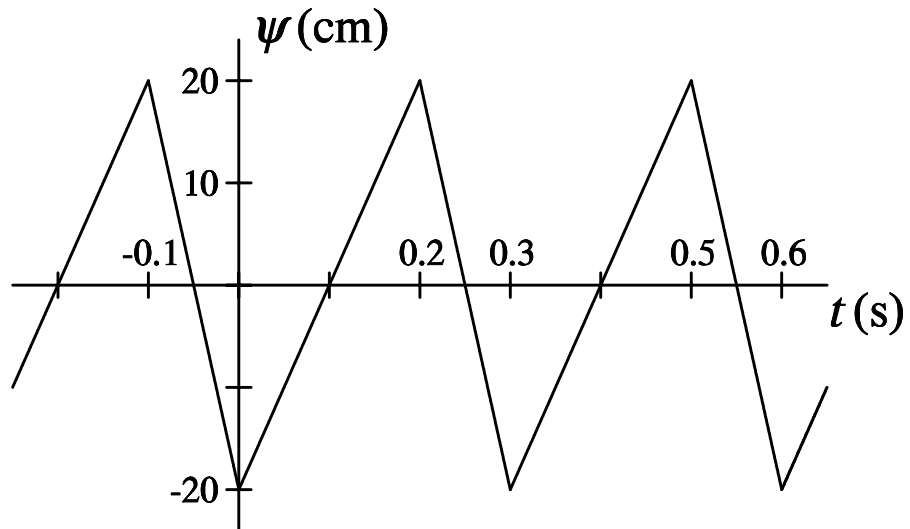
Model Solution: Homework 1A

Physics U603

Summer 1 2004

1. Hecht 2-14:

- (a) The wavelength $\lambda = 30$ cm can be read from the plot as the distance between successive maxima (or minima).
- (b) The wave cycle repeats over a period $\tau = \lambda/c = (30 \text{ cm}) / (100 \text{ cm/s}) = 0.3$ s, that is, the time it takes for the wave to travel a distance of one wavelength. Since the wave travels to the right, the transverse displacement of the rope at $z = 0$ as a function of time will look like a mirror image of the displacement as a function of position. Starting from the initial value -20 cm at time $t = 0$, the displacement of the rope will increase to $+20$ cm in 0.2 s, the time it takes for the maximum to travel 20 cm to reach $z = 0$, and then drop back to -20 cm in the remaining 0.1 s of the period. For a periodic wave, this pattern repeats indefinitely, as shown:



- (c) The frequency $\nu = 1/\tau = 3.33$ Hz is just the inverse of the period.
2. Hecht 2-17: The wavefunction has the usual form $\psi(x, t) = A \cos(kx - \omega t)$. Identification of the quantities $A = 30$ cm, $k = 6.28$ rad/m, and $\omega = 20$

rad/s allows determination of

- (a) frequency $\nu = \omega/2\pi = 3.18$ Hz
 - (b) wavelength $\lambda = 2\pi/k = 1.0$ m
 - (c) period $\tau = 1/\nu = 0.314$ s
 - (d) amplitude $A = 30$ cm
 - (e) phase velocity $c = \omega/k = 3.18$ m/s
 - (f) The negative sign in the phase indicates motion in the $+x$ direction.
3. Hecht 2-22: A wave travelling toward $-x$ has the general form $\psi(x, t) = A \cos(kx + \omega t + \phi)$. The amplitude $A = 10^3$ V/m is given, the period determines the angular frequency $\omega = 2\pi/\tau = 2.86 \times 10^{15}$ rad/s, and the speed and angular frequency determine the wave constant $k = \omega/c = 9.52 \times 10^6$ rad/m. In order to satisfy the condition $\psi(0, 0) = A$, the phase constant ϕ must satisfy $\cos \phi = 1$. The sensible choice is $\phi = 0$ (alternative choices $\phi = \pm 2\pi, \pm 4\pi, \pm 6\pi$, etc. are not wrong), so that $\psi(x, t) = 10^3$ V/m $\cos [(9.52 \times 10^6 \text{ rad/m}) x + (2.86 \times 10^{15} \text{ rad/s}) t]$.
4. Hecht 2-42: In three dimensions, a plane wave takes the general form $\psi = A \exp [i (\vec{k} \cdot \vec{r} - \omega t)]$. The wave vector \vec{k} can be written as a product of the magnitude $k = 2\pi/\lambda$, determined by the wavelength λ , with a unit vector $(4\hat{i} + 2\hat{j} + \hat{k})/\sqrt{4^2 + 2^2 + 1^2}$ that lies parallel to the direction of motion. Sclar multiplication with the position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$, allows the wavefunction to be written as

$$\psi = A \exp \left\{ i \left[\frac{k(4x + 2y + z)}{\sqrt{21}} - \omega t \right] \right\}.$$

5. Ocean waves

- (a) For large x , $\tanh x \rightarrow 1$, so for $d \gg \lambda$ the wave speed is approximately

$$c = \sqrt{\frac{g\lambda}{2\pi}}.$$

- (b) In the shallow water limit, $d \gg \lambda$, we can approximate the hyperbolic tangent

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \approx \frac{1 + x + \dots - (1 - x + \dots)}{1 + x + \dots + (1 - x + \dots)} = \frac{2x + \dots}{2 + 2x + \dots} \approx x$$

by using the Taylor series expansion of the exponentials. The resulting expression

$$c = \sqrt{\frac{g\lambda}{2\pi} \frac{2\pi d}{\lambda}} = \sqrt{gd}$$

for the wave speed no longer depends on wavelength as it does in the deep water limit.

- (c) With $\lambda = 50$ km and $d = 4$ km, the shallow water approximation already gives a value $c = \sqrt{(9.8 \text{ m/s}^2) (4 \times 10^3 \text{ m})} = 198 \text{ m/s}$, within a few percent of the result

$$c = \sqrt{\frac{(9.8 \text{ m/s}^2) (5 \times 10^4 \text{ m})}{2\pi} \tanh\left(\frac{2\pi \cdot 4}{50}\right)} = 190 \text{ m/s}$$

from the more general expression. On the open ocean, you would probably be unaware of the slow rise and fall of the water level over the $\tau = \lambda/c = (5 \times 10^4 \text{ m}) / (190 \text{ m/s}) = 263 \text{ s} = 4.4 \text{ min}$ period of the wave as it passes.

- (d) At a depth $d = 30$ m, the shallow water approximation is quite good and you can calculate the speed $c = \sqrt{(9.8 \text{ m/s}^2) (30 \text{ m})} = 17.1 \text{ m/s}$ without knowing the wavelength. This allows calculation of the wavelength $\lambda = c\tau = 4.5 \text{ km}$, since the period $\tau = 263 \text{ s}$ of the wave is unchanged. To determine the amplitude, consider that the intensity $I \propto cA^2$ measures energy transport by the wave. As the speed decreases from $c_1 = 190 \text{ m/s}$ to $c_2 = 17.1 \text{ m/s}$, the amplitude increases from $A_1 = 3 \text{ m}$ to $A_2 = A_1 (c_1/c_2)^{1/2} = 10.0 \text{ m}$ if energy is conserved.